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On the multipole electromagnetic radiation

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Abstract
After a systematic introduction of some formulae for the energy radiated by localized electric charges and currents distributions, one considers the multipole radiation and the reduction of the multipole tensors to the symmetric traceless ones. A general formula for the total power radiated by a confined system of charges is given. Although one uses Cartesian tensor components in the explicit calculations, the final results are given in a consistent tensorial form.

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1. Introduction

In the calculation of the energy radiated at large distances by a localized electric-charged system, it is not necessary to know the exact expressions of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ or of the potentials $\boldsymbol{A}$ and Φ. One may avoid the exact calculation, sometimes relatively complicated, in a simple way based on a formula for the power radiated by a charged system described by the charge ρ and current $\boldsymbol{j}$ densities with supports included in a finite domain $\mathcal{D}$ [1]:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}(\boldsymbol{\nu}, t) = \frac{r^2}{\mu_0 c}\left[\boldsymbol{\nu} \times \frac{\partial}{\partial t}\boldsymbol{A}_{\mathrm{rad}}(\boldsymbol{r}, t)\right]^2. \tag{1}$$

Here the origin O of the coordinates is chosen in the domain $\mathcal{D}$, $\boldsymbol{\nu} = \boldsymbol{r}/r$, $\mathrm{d}P/\mathrm{d}\Omega$ is related to the flow of the energy detected in the observation point $\boldsymbol{r}$ at large distance r compared with the dimensions of the given charged system. The vector $\boldsymbol{A}_{\mathrm{rad}}$ is obtained from the retarded potential

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int_{\mathcal{D}} \frac{1}{R}\boldsymbol{j}\left(\boldsymbol{r}', t - \frac{R}{c}\right)\mathrm{d}^3x'$$

with $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, by retaining only the dominant terms at large distances. A first approximation for this vector is obtained by retaining only the dominant term $1/r$ from the series expansion of $1/R$,

$$\frac{1}{R} = \frac{1}{r} + \boldsymbol{r}' \cdot \left(\nabla' \frac{1}{R}\right)_{r'=0} + \cdots = \frac{1}{r} - \boldsymbol{r}' \cdot \nabla \frac{1}{r} + \cdots = \frac{1}{r} + \frac{\boldsymbol{r}' \cdot \boldsymbol{r}}{r^3} + \cdots = \frac{1}{r} + O(1/r^2)$$

with a corresponding definition

$$\widetilde{\boldsymbol{A}}_{\mathrm{rad}}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int_{\mathcal{D}} \boldsymbol{j}\left(\boldsymbol{r}', t - \frac{R}{c}\right) \mathrm{d}^3 x'. \tag{2}$$

Supposing $r \gg \lambda$, where λ is an arbitrary wavelength from the radiation spectrum, such that the observation point is in the wave region, and retaining in equation (1) only the terms having nonzero limits for $r \to \infty$, one obtains an approximate expression related to the energy flow observed in the point $\boldsymbol{r}$ and moment t. Rigorously, this is that part of the energy flowing in the neighbourhood of the observation point which contributes to the radiated energy. In the following we assume to work in this wave region.

Equation (1) is justified in [1] using the supposed plane wave behaviour of the radiated field but also in [1] a rigorous proof is suggested for this. Indeed, this may be done by considering consistently only the terms from $\boldsymbol{E}$ and $\boldsymbol{B}$ contributing to the radiation [2, 3].

As a simple example of application of equation (1), we remember that one usually derives the angular distribution of the power radiated by a point electric charge q using the results for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ obtained from the Liénard–Wiechert potentials and retaining from the corresponding expressions only the terms contributing to the radiation [1, 2, 4]. It is very easy to calculate this distribution using in this case the formula (1), only the knowledge of the potential $\boldsymbol{A}$ being necessary [3].

In this paper we present a general procedure for expressing the multipole expansion of the power radiated by a localized system of charges. In section 2, the basic formulae for the multipole expansion of the radiation field are given. The vector potential is represented only by the symmetric and traceless multipole tensors. One uses, for this end, a reduction technique done in [7] but, this time, applying some results from [8] for giving general explicit results. In section 3, we present a general formula for the total power radiated by a confined system of charges. Some particular results are given, some of them being compared with similar results existing in the physics literature.

2. Multipolar expansion of $\mathbf{A}_{\mathrm{rad}}$ and reduction of the multipole tensors

Let us the Taylor series expansion of a function $f(R)$,

$$f(R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x'_{i_1} \cdots x'_{i_n} \partial_{i_1 \ldots i_n} f(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \boldsymbol{r}'^n \| \nabla^n f(r) \tag{3}$$

where

$$\partial_{i_1 \ldots i_n} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}}$$

and $\boldsymbol{a}^n$ is the n-fold tensorial product $(\boldsymbol{a} \otimes \cdots \otimes \boldsymbol{a})_{i_1 \ldots i_n} = a_{i_1} \cdots a_{i_n}$. Denoting by $\mathbf{T}^{(n)}$ a tensor of rank n, $\mathbf{A}^{(n)} \| \mathbf{B}^{(m)}$ is an $|n-m|$ th-rank tensor with the components

$$(\mathbf{A}^{(n)} \| \mathbf{B}^{(m)})_{i_1 \ldots i_{|n-m|}} = \begin{cases} A_{i_1 \ldots i_{n-m} j_1 \ldots j_m} B_{j_1 \ldots j_m} & n > m \\ A_{j_1 \ldots j_n} B_{j_1 \ldots j_n} & n = m \\ A_{j_1 \ldots j_n} B_{j_1 \ldots j_n i_1 \ldots i_{m-n}} & n < m. \end{cases}$$

The series expansion of the integrand from equation (2), by retaining only the $1/r$ terms contributing to the radiation, finally leads to the multipole expansion of the radiation field:

$$\boldsymbol{A}_{\text{rad}}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\left\{\sum_{n=1}^{\infty}\frac{1}{n!c^n}\left[\boldsymbol{\nu}^{n-1}\left\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathsf{M}^{(n)}(t_0)\right]\times\boldsymbol{\nu} + \sum_{n=1}^{\infty}\frac{1}{n!c^{n-1}}\boldsymbol{\nu}^{n-1}\right\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathsf{P}^{(n)}(t_0)\right\}. \tag{4}$$

Here, $\mathsf{P}^{(n)}$ and $\mathsf{M}^{(n)}$ are the electric and magnetic multipole tensors

$$\mathsf{P}^{(n)}(t) = \int_{\mathcal{D}} \boldsymbol{r}^n \rho(\boldsymbol{r},t)\,\mathrm{d}^3x \tag{5}$$

and

$$\mathsf{M}^{(n)}(t) = \frac{n}{n+1}\int_{\mathcal{D}} \boldsymbol{r}^n \times \boldsymbol{j}(\boldsymbol{r},t)\,\mathrm{d}^3x \tag{6}$$

where the 'vectorial product' $\mathsf{T}^{(n)}\times\boldsymbol{a}$ is the tensor of rank n with the components

$$(\mathsf{T}^{(n)}\times\boldsymbol{a})_{i_1\cdots i_n} = \varepsilon_{i_n i j} T_{i_1\cdots i_{n-1} i}\, a_j$$

and, particularly,

$$(\boldsymbol{b}^n\times\boldsymbol{a})_{i_1\cdots i_n} = b_{i_1}\cdots b_{i_{n-1}}(\boldsymbol{b}\times\boldsymbol{a})_{i_n}.$$

The result represented by equation (4) is obtained in [2, 7] from the series expansion of the potential $\boldsymbol{A}(\boldsymbol{r},t)$. In appendix A the series expansion (4) is justified by a straightforward expansion of $\widetilde{\boldsymbol{A}}(\boldsymbol{r},t)$.

It is possible to express $\boldsymbol{A}_{\text{rad}}$ by the reduced multipole tensors (total symmetric and traceless tensors) by applying a procedure given in [6, 7]. The special feature of this procedure consists in the possibility of representing the potentials by the series expansions obtained by simple substitutions of the multipole tensors $\mathsf{P}^{(n)}$ and $\mathsf{M}^{(n)}$, given by equations (5) and (6), by symmetric and traceless tensors $\widetilde{\mathsf{P}}^{(n)}$ and $\widetilde{\mathsf{M}}^{(n)}$ respectively. One of the advantages of this technique will be obvious in the calculation of the total power radiated by a localized system of charges.

The transformations implied by the reduction are defined such that the electromagnetic potentials $\boldsymbol{A}$ and Φ are modified only by gauge transformations implying a specific feature of the dynamic case: *the redefinitions of the multipole tensors in the lower $k < n$ orders induced by the reduction of tensors in a given order n*. In the present paper this procedure being applied to the radiation field, obviously, only the vector potential is to be considered.

The results from [7] will be applied to the radiated field in a more simple manner because of the specific feature of this problem. The reduction of multipole tensors beginning with a given order n is achieved by the following steps.

(1) The reduction of the magnetic nth-rank tensor $\mathsf{M}^{(n)}$, given by equation (6), to a symmetric tensor $\mathsf{M}_{(\text{sym})}$. Since the magnetic tensor $\mathsf{M}^{(n)}$ is symmetric only in the first $n-1$ indices, the reduction to a symmetric one may be performed by the transformation [6]

$$\begin{aligned}
\mathsf{M}_{i_1\ldots i_n} \to \mathsf{M}_{(\text{sym})i_1\ldots i_n} &= \frac{1}{n}\left[\mathsf{M}_{i_1\ldots i_n} + \mathsf{M}_{i_n i_2\ldots i_{n-1} i_1} + \cdots + \mathsf{M}_{i_1\ldots i_n i_{n-1}}\right] \\
&= \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\left[\mathsf{M}^{(\lambda)}_{i_1\ldots i_{n-1} i_\lambda i_n} - \mathsf{M}^{(\lambda)}_{i_1\ldots i_{n-1} i_n i_\lambda}\right] \\
&= \mathsf{M}_{i_1\ldots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1} q}
\end{aligned} \tag{7}$$

where we use the notations

$$\mathsf{N}_{i_1\ldots i_{n-1}} = \varepsilon_{i_{n-1}ps}\mathsf{M}_{i_1\ldots i_{n-2}ps} \qquad f^{(\lambda)}_{i_1\ldots i_n} = f_{i_1\ldots i_{\lambda-1}i_{\lambda+1}\ldots i_n}.$$

If $\mathsf{M}^{(n)}$ is given by the original definition (6), the $(n-1)$ th-rank tensor $\mathsf{N}^{(n-1)}$ is given by

$$\mathsf{N}_{i_1\ldots i_{n-1}} = \frac{n}{n+1}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-2}}[\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{j})]_{i_{n-1}}\,\mathrm{d}^3\xi.$$

We write explicitly the modification of the potential $\boldsymbol{A}_{\mathrm{rad}}$ induced by the substitution (7):

$$\begin{aligned}\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\mathrm{rad}} &\to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\mathrm{rad}} - \frac{\boldsymbol{e}_i}{n!nc^n}\varepsilon_{ikl}\nu_l\nu_{i_1}\cdots\nu_{i_{n-1}}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda kq}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathsf{N}^{(\lambda)}_{i_1\cdots i_{n-1}q}(t_0)\\ &= \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\mathrm{rad}} + \frac{n-1}{n!nc^n}\left[\boldsymbol{\nu}^{n-2}\left\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathsf{N}^{(n-1)}(t_0)\right]\right. + \frac{4\pi r}{\mu_0}\psi(\boldsymbol{r},t)\boldsymbol{\nu}\end{aligned} \tag{8}$$

where

$$\psi(\boldsymbol{r},t) = -\frac{\mu_0}{4\pi r}\frac{n-1}{n!nc^n}\left[\boldsymbol{\nu}^{n-1}\left\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathsf{N}^{(n-1)}(t_0)\right]\right.$$

and, corresponding to a gauge transformation of the potential, does not contribute to the fields $\boldsymbol{E}_{\mathrm{rad}}$ and $\boldsymbol{B}_{\mathrm{rad}}$.

(2) The *extra-gauge* alteration of the vector potential by the transformation (8) may be set off by the transformation of the electric multipole tensor $\mathsf{P}^{(n-1)}$:

$$\mathsf{P}^{(n-1)} \to \mathsf{P}'^{(n-1)} = \mathsf{P}^{(n-1)} - \frac{n-1}{c^2n^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N}^{(n-1)} \tag{9}$$

such that the final transformation of the potential is the gauge transformation

$$\boldsymbol{A}_{\mathrm{rad}} \to \boldsymbol{A}_{\mathrm{rad}} + \psi\boldsymbol{\nu}.$$

(3) After the reduction of the magnetic tensor $\mathsf{M}^{(n)}$ to a symmetric one, we have to perform the reduction to a symmetric and traceless tensor $\widetilde{\mathsf{M}}^{(n)}$. This reduction is achieved by the transformation [6]

$$\mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} \to \widetilde{\mathsf{M}}_{i_1\ldots i_n} = \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} - \sum_{D(i)}\delta_{i_1i_2}\Lambda_{i_3\ldots i_n} \tag{10}$$

where $\boldsymbol{\Lambda}^{(n-2)}$ is a symmetric tensor and the sum over $D(i)$ is the sum over all permutations of the symbols $i_1,\ldots,i_n$ which give distinct terms. Applequist [8] has given an explicit formula for expressing the components of the symmetric traceless tensor $\widetilde{\mathsf{M}}^{(n)}$ in terms of the traces of the tensor $\mathsf{M}^{(n)}$, (*the detracer theorem*, [8], equation (5.1)):

$$\widetilde{\mathsf{M}}_{i_1\cdots i_n} = \mathsf{M}_{(\mathrm{sym})i_1\ldots i_n} - \sum_{m=1}^{[n/2]}\frac{(-1)^{m-1}(2n-1-2m)!!}{(2n-1)!!}\sum_{D(i)}\delta_{i_1i_2}\cdots\delta_{i_{2m-1}i_{2m}}\mathsf{M}^{(n:m)}_{(\mathrm{sym})i_{2m+1}\ldots i_n} \tag{11}$$

where $[n/2]$ denotes the integer part of $n/2$ and $\mathsf{M}^{(n:m)}_{(\mathrm{sym})i_{2m+1}\ldots i_n}$ are the components of the $(n-2m)$ th-rank tensor obtained from $\mathsf{M}_{(\mathrm{sym})}$ by the contractions of m pairs of symbols i. Using equation (11) we may give explicitly the components of the tensor $\boldsymbol{\Lambda}^{(n-2)}$:

$$\Lambda_{i_3\ldots i_n} = \sum_{m=1}^{[n/2]}\frac{(-1)^{m-1}(2n-1-2m)!!}{(2n-1)!!\,m}\sum_{D(i)}\delta_{i_3i_4}\cdots\delta_{i_{2m-1}i_{2m}}M^{(n:m)}_{(\mathrm{sym})i_{2m+1}\ldots i_n}. \tag{12}$$

If in equations (7) and (10) $\mathsf{M}^{(n)}$ is the original one defined by equation (6), then, as is shown in [6, 2], denoting the resulting symmetric traceless tensor $\widetilde{\mathsf{M}}^{(n)}$ by $\mathcal{M}^{(n)}$, we have

$$\mathcal{M}_{i_1\ldots i_n}(t) = \frac{(-1)^{n-1}}{(n+1)(2n-1)!!}\sum_{\lambda=1}^{n}\int_{\mathcal{D}} r^{2n+1}[\boldsymbol{j}(\boldsymbol{r},t)\times\nabla)]_{i_\lambda}\partial^{(\lambda)}_{i_1\ldots i_n}\frac{1}{r}\,\mathrm{d}^3x. \tag{13}$$

In terms of the tensor $\mathbf{\Lambda}$, the modification of $\boldsymbol{A}_{\text{rad}}$ induced by the substitutions (7), (9) and (10) is obtained by a straightforward calculation

$$\begin{aligned}\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} &\to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} - \frac{\boldsymbol{e}_i}{n!c^n}\varepsilon_{ikl}\nu_l\nu_{i_1}\cdots\nu_{i_{n-1}}\sum_{D(i)}\delta_{i_1i_2}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Lambda_{i_3\ldots i_{n-1}k} \\ &= \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} - \boldsymbol{e}_i\frac{n-1}{n!c^n}\varepsilon_{ikl}\nu_l\nu_k\nu_{i_1}\cdots\nu_{i_{n-2}}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Lambda_{i_1\ldots i_{n-2}} \\ &\quad - \boldsymbol{e}_i\frac{(n-1)(n-2)}{2n!c^n}\varepsilon_{ikl}\nu_l\nu_{i_1}\cdots\nu_{i_{n-3}}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Lambda_{i_1\ldots i_{n-3}k} \\ &= \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} + \boldsymbol{e}_i\frac{(n-1)(n-2)}{2n!c^n}\varepsilon_{ilk}\nu_l\left[\boldsymbol{\nu}^{n-3}\middle\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{\Lambda}^{(n-2)}\right]_k.\end{aligned}$$

So, the transformation of the potential may be written as

$$\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} \to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} + \frac{(n-1)(n-2)}{2n!c^n}\boldsymbol{\nu}\times\left[\boldsymbol{\nu}^{n-3}\middle\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{\Lambda}^{(n-2)}\right].$$

(4) It is a simple matter to see that the last *extra-gauge* term may be set off by the transformation

$$\mathsf{M}^{(n-2)} \to \mathsf{M}'^{(n-2)} = \mathsf{M}^{(n-2)} + \frac{n-2}{2c^2n}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{\Lambda}^{(n-2)}. \tag{14}$$

(5) This step consists in the reduction of the symmetric nth-order electric multipole tensor $\mathsf{P}^{(n)}$ to a symmetric and traceless one by a transformation of the type (10):

$$\mathsf{P}_{i_1\ldots i_n} \to \widetilde{\mathsf{P}}_{i_1\ldots i_n} = \mathsf{P}_{i_1\ldots i_n} - \sum_{D(i)}\delta_{i_1i_2}\Pi_{i_3\ldots i_n} \tag{15}$$

where the symmetric tensor $\mathbf{\Pi}^{(n-2)}$ is defined in terms of the traces of the tensor $\mathsf{P}^{(n)}$ by a relation similar to equation (12).

If in equation (15) $\mathsf{P}^{(n)}$ is the original one given by equation (5) then the resulting symmetric traceless tensor $\widetilde{\mathsf{P}}^{(n)}$ is denoted by $\mathcal{P}^{(n)}$ and we have [9]

$$\mathcal{P}_{i_1\ldots i_n} = \frac{(-1)^n}{(2n-1)!!}\int_{\mathcal{D}}\rho(\boldsymbol{r},t)r^{2n+1}\nabla^n\frac{1}{r}\,\mathrm{d}^3x. \tag{16}$$

The components of the kth-rank tensor $\nabla^k(1/r)$ in equations (13) and (16) are spherical harmonics of degree $-k-1$ [8–10].

The resulting transformation of $\boldsymbol{A}_{\text{rad}}$ is

$$\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} \to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} - \frac{\boldsymbol{e}_i}{n!c^{n-1}}\nu_{i_1}\cdots\nu_{i_{n-1}}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\sum_{D(i)}\delta_{i_1i_2}\Pi_{i_3\ldots i_{n-1}i}.$$

In this equation, we have $(n-1)$ terms with δ_{ik}, $k = 1,\ldots,n-1$ and $(n-1)(n-2)/2$ terms with $\delta_{i_ji_k}$, $j,k = 1,2,\ldots,n-1$ so that

$$\begin{aligned}\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} \to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} &+ \frac{4\pi r}{\mu_0}\psi\boldsymbol{\nu} - \frac{n-1}{n!c^{n-1}}\boldsymbol{\nu}\nu_{i_2}\cdots\nu_{i_{n-1}}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Pi_{i_2\ldots i_{n-1}} \\ &- \frac{(n-1)(n-2)}{2n!c^{n-1}}\nu_{i_3}\cdots\nu_{i_{n-1}}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Pi_{i_3\ldots i_{n-1}i}\end{aligned}$$

that is

$$\frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} \to \frac{4\pi r}{\mu_0}\boldsymbol{A}_{\text{rad}} + \frac{4\pi r}{\mu_0}(\psi+\psi')\boldsymbol{\nu} - \frac{(n-1)(n-2)}{2n!c^{n-1}}\left[\nu^{n-3}\middle\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathbf{\Pi}^{(n-2)}\right] \tag{17}$$

where

$$\psi' = -\frac{\mu_0}{4\pi r}\frac{(n-1)}{n!c^{n-1}}\boldsymbol{\nu}^{n-2}\left\|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\boldsymbol{\Pi}^{(n-2)}.\right.$$

(6) The alteration of the potential represented by the last term in equation (17) is set off by the transformation

$$\mathsf{P}^{(n-2)} \to \mathsf{P}^{(n-2)} + \frac{n-2}{2nc^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\Pi}^{(n-2)} \tag{18}$$

which preserves the symmetry properties of $\mathsf{P}^{(n-2)}$.

By this last transformation (18), the reduction of the multipole tensors in the given nth order is achieved. Now, to carry out this procedure to the $(n-1)$th order, we must realize that in this order some tensors have been already modified in order to set off the alterations of the electromagnetic field by the reductions in the nth order. So, the transformation (9) alters the symmetry properties of the $(n-1)$th-order electric multipole tensor because

$$\delta\mathsf{P}^{(n-1)} = -\frac{n-1}{c^2n^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N}^{(n-1)}(t_0)$$

is symmetric only in the first $n-2$ indices. To restore the full symmetry of the $(n-1)$th-order electric moment, we perform the reduction of $\mathsf{N}^{(n-1)}$ to a symmetric tensor by the transformation

$$\mathsf{N}_{i_1\ldots i_{n-1}} \to \mathsf{N}_{i_1\ldots i_{n-1}} \to \mathsf{N}_{i_1\ldots i_{n-1}} - \frac{1}{n-1}\sum_{\lambda=1}^{(n-2)}\left[\mathsf{N}_{i_1\ldots i_{n-1}}\mathsf{N}^{(\lambda)}_{i_1\ldots i_{n-1}i_\lambda}\right]. \tag{19}$$

By introducing the tensor $\mathcal{N}^{(n-2)}$ with the components

$$\mathcal{N}_{i_1\ldots i_{n-2}} = \varepsilon_{i_{n-2}ps}\mathsf{N}_{i_1\ldots i_{n-3}ps}$$

the transformation (19) may be written as

$$\mathsf{N}_{i_1\ldots i_{n-1}} \to \mathsf{N}_{i_1\ldots i_{n-1}} - \frac{1}{n-1}\sum_{\lambda=1}^{(n-2)}\varepsilon_{i_\lambda i_{n-1}q}\mathcal{N}^{(\lambda)}_{i_1\ldots i_{n-2}q}$$

and

$$\begin{aligned}\mathsf{P}'_{i_1\ldots i_{n-1}} &= \mathsf{P}_{i_1\ldots i_{n-1}} - \frac{n-1}{c^2n^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N}_{i_1\ldots i_{n-1}} \\ &\to \mathsf{P}_{i_1\ldots i_{n-1}} - \frac{n-1}{c^2n^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{N}_{i_1\ldots i_{n-1}} + \frac{1}{c^2n^2}\sum_{\lambda=1}^{(n-2)}\varepsilon_{i_\lambda i_{n-1}q}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}^{(\lambda)}_{i_1\ldots i_{n-2}q}.\end{aligned} \tag{20}$$

If $\mathsf{M}^{(n)}$ is given by the original definition (6), then we can write

$$\mathcal{N}_{i_1\ldots i_{n-2}} = -\frac{n}{n+1}\int_{\mathcal{D}}\xi^2\xi_{i_1}\cdots\xi_{i_{n-3}}\,(\boldsymbol{\xi}\times\boldsymbol{j})_{i_{n-2}}\,\mathrm{d}^3\xi.$$

The alteration of the vector potential $\boldsymbol{A}_{\mathrm{rad}}$ by the transformation (20) is given by

$$\boldsymbol{A}_{\mathrm{rad}} \to \boldsymbol{A}_{\mathrm{rad}} - \frac{n-2}{n!nc^n}\boldsymbol{\nu}\times\left[\boldsymbol{\nu}^{n-3}\times\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{N}}^{(n-2)}(t_0)\right].$$

This alteration of $\boldsymbol{A}_{\mathrm{rad}}$ is set off by the transformation of $\mathsf{M}'^{(n-2)}$, given by equation (14),

$$\mathsf{M}'^{(n-2)} \to \mathsf{M}''^{(n-2)} = \mathsf{M}'^{(n-2)} - \frac{n-2}{n^2(n-1)c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\mathcal{N}}^{(n-2)}. \tag{21}$$

By this transformation the symmetry properties of $\mathsf{M}'^{(n-2)}$ are preserved. Particularly, by reducing the $(n-2)$ th-rank multipole tensors, in the case of $\mathsf{M}''^{(n-2)}$, we have to obtain only the symmetric part of the supplementary term from equation (14).

If we begin the reduction from a given order n, then the results of the reductions of $\mathsf{P}^{(n)}$ and $\mathsf{M}^{(n)}$ are the tensors $\boldsymbol{\mathcal{P}}^{(n)}$ and $\boldsymbol{\mathcal{M}}^{(n)}$ given by equations (16) and (13) but for $k < n$ the kth-order reduced multipole tensors may differ from $\boldsymbol{\mathcal{P}}^{(k)}$ and $\boldsymbol{\mathcal{M}}^{(k)}$ by terms induced by the procedure of the reductions from the previous steps. These last terms give contributions to the potentials and fields expressed by toroidal moments and mean radii of various orders.

We give here some simple examples of reductions, and we will see how naturally the toroidal moments appear as a result of such an approach.

We consider in the following the reduction of the magnetic and electric multipole tensors beginning from the μth and εth orders, respectively (generally, considering multipole's contributions of the same orders, $\mu = \varepsilon - 1$).

For $(\mu, \varepsilon) = (1, 2)$, we have $\mathsf{M}^{(1)} \to \widetilde{\mathsf{M}}^{(1)} = \mathsf{M}^{(1)}, \mathsf{P}^{(1)} \to \widetilde{\mathsf{P}}^{(1)} = \mathsf{P}^{(1)}, \mathsf{P}^{(2)} \to \widetilde{\mathsf{P}}^{(2)} = \mathcal{P}^{(2)}$. These transformations produce only a gauge transformation of $\boldsymbol{A}_{\mathrm{rad}}$.

We give in appendix a scheme of the reductions for $(\mu, \varepsilon) = (4, 5)$ from which we can obtain also the cases $(\mu, \varepsilon) = (2, 3)$ and $(\mu, \varepsilon) = (3, 4)$. In [3] is also given the scheme of reductions corresponding to the case $(\mu, \varepsilon) = (5, 6)$. For arbitrary μ and ε, we think it is possible to find a general rule or, at least, to elaborate symbolic computer programs.

In the case $(\mu, \varepsilon) = (2, 3)$,

$$\mathsf{M}^{(k)} \longrightarrow \widetilde{\mathsf{M}}^{(k)} = \mathcal{M}^{(k)} \qquad k = 1, 2 \qquad \mathsf{P}^{(1)} \to \widetilde{\mathsf{P}}^{(1)} = \mathsf{P}^{(1)} - \frac{1}{4c^2}\dot{\mathsf{N}}^{(1)} + \frac{1}{6c^2}\ddot{\Pi}^{(1)}$$

$$\mathsf{P}^{(k)} \longrightarrow \widetilde{\mathsf{P}}^{(k)} = \mathcal{P}^{(k)} \qquad k = 2, 3.$$

Here, $\mathsf{N}^{(1)}$ and $\Pi^{(1)}$ are given by equations (B.3) and (B.8) for $\mathsf{N}_{qqi} = 0, \mathsf{P}_{qqppi} = 0$, that is eliminating the contributions from the orders $n_\mu > 2$ of the magnetic multipole tensors and from the orders $n_\varepsilon > 3$ for the electric ones.

Taking into account the continuity equation verified by ρ and $\boldsymbol{j}$, we obtain

$$\widetilde{\mathsf{P}}_i = \mathsf{P}_i - \frac{1}{c^2}\dot{\mathsf{T}}_i \qquad \mathsf{T}_i = \frac{1}{10}\int_{\mathcal{D}} [(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i - 2\xi^2 j_i]\,\mathrm{d}^3\xi \tag{22}$$

where T_i are the Cartesian components of the toroid dipole tensor [11–13].

In the case $(\mu, \varepsilon) = (3, 4)$, we have the changes

$$\mathsf{M}^{(1)} \to \widetilde{\mathsf{M}}^{(1)} = \mathsf{M}^{(1)} + \frac{1}{c^2}\ddot{\Lambda}^{(1)} \qquad \mathsf{M}^{(k)} \to \widetilde{\mathsf{M}}^{(k)} = \mathcal{M}^{(k)} \qquad k = 2, 3$$

$$\mathsf{P}^{(1)} \to \widetilde{\mathsf{P}}^{(1)} = \mathsf{P}^{(1)} - \frac{1}{4c^2}\dot{\mathsf{N}}^{(1)} + \frac{1}{6c^2}\ddot{\Pi}^{(1)}$$

$$\mathsf{P}^{(2)} \to \widetilde{\mathsf{P}}^{(2)} = \mathcal{P}^{(2)} - \frac{2}{9c^2}\dot{\widetilde{\mathsf{N}}}^{(2)} + \frac{1}{4c^2}\ddot{\Pi}^{(2)} \qquad \mathsf{P}^{(k)} \to \widetilde{\mathsf{P}}^{(k)} = \mathcal{P}^{(k)} \qquad k = 3, 4$$

where $\widetilde{\mathsf{N}}_{ij}, \Lambda_i, \mathsf{N}_{ij}$ and Π_{ij} are given by equations (B.2), (B.3), (B.5), (B.6) and (B.9) by eliminating the contributions from the orders $n_\mu > 3$ and $n_\varepsilon > 4$. In this case, one obtains the contribution of the toroidal quadrupole tensor $\mathsf{T}^{(2)}$ having the Cartesian components [11–14]:

$$\mathsf{T}_{ik} = \frac{1}{42}\int_{\mathcal{D}} [4(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i\xi_k - 5\xi^2(\xi_i j_k + \xi_k j_i) + 2\xi^2(\boldsymbol{\xi}\cdot\boldsymbol{j})\delta_{ik}]\,\mathrm{d}^3\xi$$

having, besides equation (22),

$$\widetilde{\mathsf{P}}_{ik} = \mathcal{P}_{ik} - \frac{1}{c^2}\dot{\mathsf{T}}_{ik}$$

and the dipolar magnetic moment modified by a mean-square current radius

$$\widetilde{\mathsf{M}}_i = \mathsf{M}_i + \frac{1}{c^2}\frac{1}{20}\int_{\mathcal{D}} \xi^2(\boldsymbol{\xi}\times\boldsymbol{j})\,\mathrm{d}^3\xi.$$

In the case $(\mu, \varepsilon) = (4, 5)$, we point out the result for $\mathsf{P}^{(3)}$

$$\widetilde{\mathsf{P}}^{(3)} = \mathcal{P}^{(3)} - \frac{3}{16c^2}\ddot{\widetilde{\mathsf{N}}}^{(3)} + \frac{3}{10c^2}\ddot{\widetilde{\Pi}}^{(3)} : \widetilde{\mathsf{P}}_{ijk} = \mathcal{P}_{ijk} - \frac{1}{c^2}\dot{\mathsf{T}}_{ijk}$$

with the toroidal moment

$$\mathsf{T}_{ijk} = \frac{1}{60}\int_{\mathcal{D}}\left[\xi^4\sum_{D(i,j,k)}\delta_{ij}j_k + \xi^2(\boldsymbol{\xi}\cdot\boldsymbol{j})\sum_{D(i,j,k)}\delta_{ij}\xi_k + 5(\boldsymbol{\xi}\cdot\boldsymbol{j})\xi_i\xi_j\xi_k - 5\xi^2\sum_{D(i,j,k)}\xi_i\xi_j j_k\right]\mathrm{d}^3\xi.$$

Other explicit results in this case are given in [3].

These results show that one may obtain from the formula (4) the correct representation of the electromagnetic field by the reduced multipole tensors, but introducing these tensors up to a given order n, we obtain separate contributions from some electric toroidal moments and mean $2n$-power radii. This was pointed out firstly by Dubovik *et al* [11–13]. In the present paper we point out that considering the contributions to the electromagnetic field of some toroidal moments, one suppose the reduction of the multipole tensors up to a well-defined maximal order n.

3. The total power radiated by a localized system of charges

Let us the total radiation power obtained by integrating equation (1):

$$\mathcal{I}_{\mu,\varepsilon} = \frac{1}{\mu_0 c}\int\left(\boldsymbol{\nu}\times\frac{\partial\boldsymbol{A}_{\mathrm{rad}}}{\partial t}\right)^2_{\mu,e} r^2\,\mathrm{d}\Omega(\boldsymbol{\nu}) \tag{23}$$

considering only the contributions of the magnetic and electric multipoles up to the μth and εth orders, respectively. Using equation (4) with the substitution $\mathsf{P}^{(k)} \longrightarrow \widetilde{\mathsf{P}}^{(k)}, k \leqslant \varepsilon$ and $\mathsf{M}^{(l)} \longrightarrow \widetilde{\mathsf{M}}^{(l)}, l \leqslant \mu$ we may write

$$\begin{aligned}\left(\frac{4\pi r}{\mu_0}\right)^2(\boldsymbol{\nu}\times\boldsymbol{A}_{\mathrm{rad}})^2_{\mu,e} &= \sum_{n=1}^{\mu}\sum_{m=1}^{\mu}\frac{1}{n!m!c^{n+m}}\left[\left(\boldsymbol{\nu}^{n-1}\|\widetilde{\mathsf{M}}^{(n)}_{,n}\right)\cdot\left(\boldsymbol{\nu}^{m-1}\|\widetilde{\mathsf{M}}^{(m)}_{,m}\right) - \left(\boldsymbol{\nu}^{n}\|\widetilde{\mathsf{M}}^{(n)}_{,n}\right)\left(\boldsymbol{\nu}^{m}\|\widetilde{\mathsf{M}}^{(m)}_{,m}\right)\right]\\ &+ \sum_{n=1}^{\varepsilon}\sum_{m=1}^{\varepsilon}\frac{1}{n!m!c^{n+m-2}}\left[\left(\boldsymbol{\nu}^{n-1}\|\widetilde{\mathsf{P}}^{(n)}_{,n}\right)\cdot\left(\boldsymbol{\nu}^{m-1}\|\widetilde{\mathsf{P}}^{(m)}_{,m}\right) - \left(\boldsymbol{\nu}^{n}\|\widetilde{\mathsf{P}}^{(n)}_{,n}\right)\left(\boldsymbol{\nu}^{m}\|\widetilde{\mathsf{P}}^{(m)}_{,m}\right)\right]\\ &+ 2\sum_{n=1}^{\mu}\sum_{m=1}^{\varepsilon}\frac{1}{n!m!c^{n+m-1}}\left\{\left(\boldsymbol{\nu}^{n-1}\|\widetilde{\mathsf{M}}^{(n)}_{,n}\right)\cdot\left[\boldsymbol{\nu}\times\left(\boldsymbol{\nu}^{m-1}\|\widetilde{\mathsf{P}}^{(m)}_{,m}\right)\right]\right\}\end{aligned} \tag{24}$$

where

$$\mathsf{T}^{(n)}_{,k} = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathsf{T}^{(n)}.$$

The calculation of the integrals in equation (23) is reduced to the calculation of $\left\langle \nu_{i_1}\cdots\nu_{i_n}\right\rangle_{\boldsymbol{\nu}}$, $n=0,1,\ldots$ with

$$\langle f(\boldsymbol{\nu})\rangle_{\boldsymbol{\nu}} = \frac{1}{4\pi}\int f(\boldsymbol{\nu})\,\mathrm{d}\Omega(\boldsymbol{\nu}):$$

$$\left\langle \nu_{i_1}\cdots\nu_{i_{2n+1}}\right\rangle_{\boldsymbol{\nu}} = 0 \qquad \left\langle \nu_{i_1}\cdots\nu_{i_{2n}}\right\rangle_{\boldsymbol{\nu}} = \frac{1}{(2n+1)!!}\sum_{D(i)}\delta_{i_1 i_2}\cdots\delta_{i_{2n-1}i_{2n}}.$$

Let us the symmetric and traceless tensors $\mathsf{A}^{(n)}$ and $\mathsf{B}^{(m)}$ and the averaged contraction

$$\langle(\boldsymbol{\nu}^k\|\mathsf{A}^{(n)})\|(\boldsymbol{\nu}^{k'}\|\mathsf{B}^{(m)})\rangle_{\boldsymbol{\nu}} = \left\langle \nu_{i_1}\cdots\nu_{i_k}\nu_{j_1}\cdots\nu_{j_{k'}}\mathsf{A}_{i_1\ldots i_k i_{k+1}\ldots i_n}B_{j_1\ldots j_{k'}j_{k'+1}\ldots j_m}\right\rangle_{\boldsymbol{\nu}}.$$

This is non-zero only for the products of $\delta_{i_p j_q}$ with $p=1,\ldots k$, $q=1,\ldots k'$, and it is easy to demonstrate the relation

$$\langle(\boldsymbol{\nu}^k\|\mathsf{A}^{(n)})\|(\boldsymbol{\nu}^{k'}\|\mathsf{B}^{(m)})\rangle_{\boldsymbol{\nu}} = \frac{k!}{(2k+1)!!}[\mathsf{A}^{(n)}\|\mathsf{B}^{(m)}]\delta_{k'k}.$$

The terms of the last sum from equation (24) give contributions to the total radiated power of the form

$$\left\langle \nu_{i_1}\cdots\nu_{i_{n-1}}\nu_{j_1}\cdots\nu_{j_{m-1}}\nu_p\right\rangle\varepsilon_{i_n pq}\mathsf{A}_{i_1\ldots i_n}\mathsf{B}_{j_1\ldots j_{m-1}q}$$

but all the terms from the sum of δ-products representing the averaged products of ν contains either $\delta_{i_k p}$ or $\delta_{p j_l}$, $k=1,\ldots,n-1$, $l=1,\ldots,m-1$ such that, because of $\varepsilon_{i_n pq}$ and of the traceless character of A and B, the result is zero. Using these results in equations (23) and (24), we obtain

$$\mathcal{I}_{\mu,\varepsilon} = \frac{1}{4\pi\varepsilon_0 c^3}\left[\sum_{n=1}^{\mu}\frac{n+1}{nn!(2n+1)!!}\frac{1}{c^{2n}}\left[\widetilde{\mathsf{M}}^{(n)}_{,n+1}\middle\|\widetilde{\mathsf{M}}^{(n)}_{,n+1}\right] + \sum_{n=1}^{\varepsilon}\frac{n+1}{nn!(2n+1)!!}\frac{1}{c^{2n-2}}\left[\widetilde{\mathsf{P}}^{(n)}_{,n+1}\middle\|\widetilde{\mathsf{P}}^{(n)}_{,n+1}\right]\right]. \tag{25}$$

For comparison with results existing in the literature [1, 11–13] we write here the results in the following cases. The case $(\mu,\varepsilon)=(1,2)$ is given in [1]

$$\begin{aligned}\mathcal{I}_{1,2} &= \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\widetilde{\mathsf{P}}^{(1)}_{,2}\middle\|\widetilde{\mathsf{P}}^{(1)}_{,2} + \frac{1}{20c^2}\widetilde{\mathsf{P}}^{(2)}_{,3}\middle\|\widetilde{\mathsf{P}}^{(2)}_{,3} + \frac{2}{3c^2}\widetilde{\mathsf{M}}^{(1)}_{,2}\middle\|\widetilde{\mathsf{M}}^{(1)}_{,2}\right]\\ &= \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\ddot{\boldsymbol{p}}^2 + \frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\dddot{\mathcal{P}}^{(2)}\middle\|\dddot{\mathcal{P}}^{(2)}\right].\end{aligned} \tag{26}$$

This result is justified by the invariance of the radiation field to the transformation $\mathsf{P}^{(2)}\to\mathcal{P}^{(2)}$.

In the case $(\mu,\varepsilon)=(2,3)$ we obtain

$$\begin{aligned}\mathcal{I}_{2,3} = \frac{1}{4\pi\varepsilon_0 c^3}&\left[\frac{2}{3}\widetilde{\mathsf{P}}^{(1)}_{,2}\middle\|\widetilde{\mathsf{P}}^{(1)}_{,2} + \frac{1}{20c^2}\widetilde{\mathsf{P}}^{(2)}_{,3}\middle\|\widetilde{\mathsf{P}}^{(2)}_{,3} + \frac{2}{945c^4}\widetilde{\mathsf{P}}^{(3)}_{,4}\middle\|\widetilde{\mathsf{P}}^{(3)}_{,4}\right.\\ &\left.+\frac{2}{3c^2}\widetilde{\mathsf{M}}^{(1)}_{,2}\middle\|\widetilde{\mathsf{M}}^{(1)}_{,2} + \frac{1}{20c^4}\widetilde{\mathsf{M}}^{(2)}_{,3}\middle\|\widetilde{\mathsf{M}}^{(2)}_{,3}\right] = \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\left|\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathsf{T}}\right|^2\right.\\ &\left.+\frac{2}{3c^2}\ddot{\boldsymbol{m}}^2 + \frac{1}{20c^2}\dddot{\mathcal{P}}^{(2)}\middle\|\dddot{\mathcal{P}}^{(2)} + \frac{1}{20c^4}\dddot{\mathcal{M}}^{(2)}\middle\|\dddot{\mathcal{M}}^{(2)} + \frac{2}{945c^4}\mathcal{P}^{(3)}_{,4}\middle\|\mathcal{P}^{(3)}_{,4}\right].\end{aligned} \tag{27}$$

In the case $(\mu, \varepsilon) = (3, 4)$,

$$\mathcal{I}_{3,4} = \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\widetilde{\mathsf{P}}^{(1)}_{,2}\Big\|\widetilde{\mathsf{P}}^{(1)}_{,2} + \frac{1}{20c^2}\widetilde{\mathsf{P}}^{(2)}_{,3}\Big\|\widetilde{\mathsf{P}}^{(2)}_{,3} + \frac{2}{945c^4}\widetilde{\mathsf{P}}^{(3)}_{,4}\Big\|\widetilde{\mathsf{P}}^{(3)}_{,4} + \frac{1}{18144c^6}\widetilde{\mathsf{P}}^{(4)}_{,5}\Big\|\widetilde{\mathsf{P}}^{(4)}_{,5}\right.$$
$$\left. + \frac{2}{3c^2}\widetilde{\mathsf{M}}^{(1)}_{,2}\Big\|\widetilde{\mathsf{M}}^{(1)}_{,2} + \frac{1}{20c^4}\widetilde{\mathsf{M}}^{(2)}_{,3}\Big\|\widetilde{\mathsf{M}}^{(2)}_{,3} + \frac{2}{945c^4}\widetilde{\mathsf{M}}^{(3)}_{,4}\Big\|\widetilde{\mathsf{M}}^{(3)}_{,4}\right]$$
$$= \frac{1}{4\pi\varepsilon_0 c^3}\left[\frac{2}{3}\left(\ddot{\boldsymbol{p}} - \frac{1}{c^2}\dddot{\mathsf{T}}\right)^2 + \frac{2}{3c^2}\left(\ddot{\boldsymbol{m}} + \frac{1}{c^2}\Lambda_{,4}\right)^2\right.$$
$$+ \frac{1}{20c^2}\left(\dddot{\mathcal{P}}^{(2)} - \frac{1}{c^2}\mathsf{T}^{(2)}_{,4}\right)\Big\|\left(\dddot{\mathcal{P}}^{(2)} - \frac{1}{c^2}\mathsf{T}^{(2)}_{,4}\right) + \frac{1}{20c^4}\dddot{\mathcal{M}}^{(2)}\Big\|\dddot{\mathcal{M}}^{(2)}$$
$$\left. + \frac{2}{945c^4}\left(\mathcal{P}^{(3)}_{,4}\Big\|\mathcal{P}^{(3)}_{,4} + \frac{1}{c^2}\mathcal{M}^{(3)}_{,4}\Big\|\mathcal{M}^{(3)}_{,4}\right) + \frac{1}{18144c^6}\mathcal{P}^{(4)}_{,5}\Big\|\mathcal{P}^{(4)}_{,5}\right]. \tag{28}$$

4. Conclusion

In the physics literature on multipole expansions, the tensorial notation was abandoned practically in favour of the spherical functions [9]. On the other hand, in many applications, it is relatively easy to apply the tensor formalism as was shown, for example, in [9]. In the present paper, calculation of the total power radiated by a confined system of charges is also such an example. Starting from equations written with tensorial notations, it is easy to materialize the corresponding expressions in the Cartesian or spherical components because in the literature explicit formulae are derived which relate Cartesian and spherical components of the irreducible Cartesian tensors [15, 16].

Appendix A. Multipolar expansion of $\boldsymbol{A}_{\text{rad}}$

By introducing the expansion (3) into equation (2), we obtain

$$\widetilde{\boldsymbol{A}}_{\text{rad}}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi r}\boldsymbol{e}_i\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\partial_{i_1\ldots i_n}\int_{\mathcal{D}} x'_{i_1}\cdots x'_{i_n} j_i(\boldsymbol{r}', t_0)\,\mathrm{d}^3x' = \frac{\mu_0}{4\pi r}\boldsymbol{e}_i\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_i^{(n)} \tag{A.1}$$

where $t_0 = t - r/c$ and,

$$a_i^{(n)} = \partial_{i_1\ldots i_n}\int_{\mathcal{D}} x'_{i_1}\cdots x'_{i_n} j_i\left(\boldsymbol{r}', t - \frac{r}{c}\right)\mathrm{d}^3x' \tag{A.2}$$

and $\boldsymbol{e}_i$ are the orthogonal unit vectors along the axes.

In the following, we use a generalization to the dynamic case of a procedure given in [4, 5] in the magnetostatic case. Let the identity

$$\boldsymbol{\nabla}[x_i\boldsymbol{j}(\boldsymbol{r}, t)] = j_i(\boldsymbol{r}, t) + x_i\boldsymbol{\nabla}\boldsymbol{j}(\boldsymbol{r}, t).$$

Considering the continuity equation $\boldsymbol{\nabla}\boldsymbol{j} + \partial\rho/\partial t = 0$, we may write

$$j_i(\boldsymbol{r}, t) = \boldsymbol{\nabla}[x_i\boldsymbol{j}(\boldsymbol{r}, t)] + x_i\frac{\partial}{\partial t}\rho(\boldsymbol{r}, t)$$

and using this last equation in equation (A.2), we get

$$a_i^{(n)} = -\partial_{i_1\ldots i_n} \int_{\mathcal{D}} x_i' \boldsymbol{j}(\boldsymbol{r}', t_0) \cdot \nabla'(x_{i_1}' \cdots x_{i_n}')\, \mathrm{d}^3x' + \partial_{i_1\ldots i_n} \int_{\mathcal{D}} x_{i_1}' \cdots x_{i_n}' x_i' \frac{\partial}{\partial t} \rho(\boldsymbol{r}', t_0)\, \mathrm{d}^3x'$$

considering a null surface term because $\boldsymbol{j} = 0$ on $\partial\mathcal{D}$. Because of the symmetry of the derivative tensor and introducing the nth-order electric multipole tensor, we may write

$$\begin{aligned} a_i^{(n)} &= -n\partial_{i_1\ldots i_n} \int_{\mathcal{D}} x_{i_1}' \cdots x_{i_{n-1}}' x_i' j_{i_n}(\boldsymbol{r}', t_0)\, \mathrm{d}^3x' + \left[\nabla^n \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}^{(n+1)}(t_0)\right.\right]_i \\ &= -n\partial_{i_1\ldots i_n} \int_{\mathcal{D}} x_{i_1}' \cdots x_{i_{n-1}}' (x_i' j_{i_n} - x_{i_n}' j_i)\, \mathrm{d}^3x' - n\partial_{i_1\ldots i_n} \int_{\mathcal{D}} x_{i_1}' \cdots x_{i_n}' j_i\, \mathrm{d}^3x' \\ &\quad + \left[\nabla^n \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}^{(n+1)}(t_0)\right.\right]_i, \end{aligned}$$

that is

$$a_i^{(n)} = -\frac{n\varepsilon_{kii_n}}{n+1} \partial_{i_n} \partial_{i_1\ldots i_{n-1}} \int_{\mathcal{D}} x_{i_1}' \cdots x_{i_{n-1}}' (\boldsymbol{r}' \times \boldsymbol{j})_k\, \mathrm{d}^3x' + \frac{1}{n+1} \left[\nabla^n \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}^{(n+1)}(t_0)\right.\right]_i. \tag{A.3}$$

We may use in equation (A.3) the definition (6) of the nth-order magnetic multipole momentum given in [5]. So, equation (A.3) may be written as

$$\boldsymbol{a}^{(n)} = -\nabla \times (\nabla^{n-1} \| \mathsf{M}^{(n)}(t_0)) + \frac{1}{n+1} \nabla^n \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{P}^{(n+1)}(t_0). \right.$$

Going back to the expansion (A.1), we may write

$$\widetilde{\boldsymbol{A}}_{\mathrm{rad}}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi r} \nabla \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left\| \mathsf{M}^{(n)}(t_0) \right. + \frac{\mu_0}{4\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left\| \frac{\partial}{\partial t} \mathsf{P}^{(n)}(t_0). \right.$$

Now, we extract from the last equation the terms contributing to the radiation. Because

$$\partial_{i_1\ldots i_n} f\left(t - \frac{r}{c}\right) = \frac{(-1)^n}{c^n} \nu_{i_1} \cdots \nu_{i_n} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f\left(t - \frac{r}{c}\right) + O\left(\frac{1}{r}\right)$$

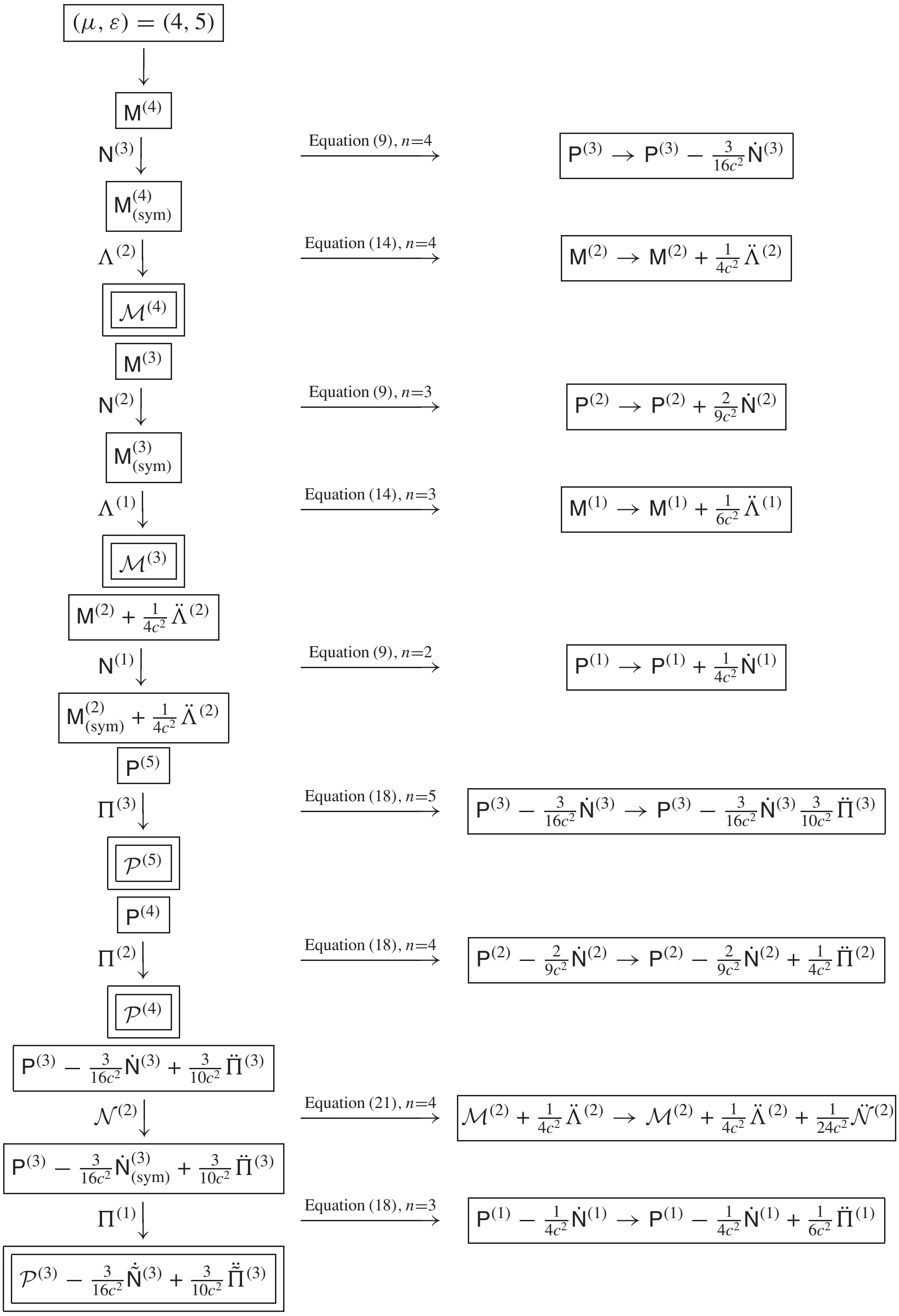
we have

$$\begin{aligned} \nabla \times \left[\nabla^{n-1} \left\| \mathsf{M}^{(n)}\left(t - \frac{r}{c}\right)\right.\right] &= \frac{(-1)^n}{c^n} \boldsymbol{e}_i \varepsilon_{ijk} \nu_j \nu_{i_1} \cdots \nu_{i_{n-1}} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathsf{M}_{i_1\ldots i_{n-1}k}\left(t - \frac{r}{c}\right) + O\left(\frac{1}{r}\right) \\ &= \frac{(-1)^{n-1}}{c^n} \left[\boldsymbol{\nu}^{n-1} \left\| \frac{\mathrm{d}^n}{\mathrm{d}t^n} \mathsf{M}^{(n)}\left(t - \frac{r}{c}\right)\right.\right] \times \boldsymbol{\nu} + O\left(\frac{1}{r}\right) \\ &= \frac{(-1)^{n-1}}{c^n} \boldsymbol{\nu}^{n-1} \left\| \left[\frac{\mathrm{d}^n \mathsf{M}^{(n)}(t - r/c)}{\mathrm{d}t^n} \times \boldsymbol{\nu}\right] \right. + O\left(\frac{1}{r}\right) \end{aligned}$$

obtaining, finally, the formula (4), giving the explicit contribution of each multipole to the radiation field.

Appendix B. Reduction of multipole tensors

We give a scheme of the reductions of multipole tensors for $(\mu, \varepsilon) = (4, 5)$:



$$\boxed{\mathsf{P}^{(2)} - \tfrac{2}{9c^2}\dot{\mathsf{N}}^{(2)} + \tfrac{1}{4c^2}\ddot{\Pi}^{(2)}}$$

$$\mathcal{N}^{(1)}\big\downarrow \qquad\qquad \xrightarrow{\text{Equation (21)},\ n=3} \qquad \boxed{\mathsf{M}^{(1)}_{\cdot\cdot} \to \boxed{\mathsf{M}^{(1)} + \tfrac{1}{6c^2}\ddot{\Lambda}^{(1)} - \tfrac{1}{18c^2}\ddot{\mathcal{N}}^{(1)}}}$$

$$\boxed{\mathsf{P}^{(2)} - \tfrac{2}{9c^2}\dot{\mathsf{N}}^{(2)}_{\mathrm{sym})} + \tfrac{1}{4c^2}\ddot{\Pi}^{(2)}}$$

$$\Pi\big\downarrow$$

$$\boxed{\boxed{\mathcal{P}^{(2)} - \tfrac{2}{9c^2}\dot{\widetilde{\mathsf{N}}}^{(2)} + \tfrac{1}{4c^2}\ddot{\widetilde{\Pi}}^{(2)}}}$$

$$\boxed{\mathcal{M}^{(2)} + \tfrac{1}{4c^2}\ddot{\Lambda}^{(2)} - \tfrac{1}{24c^2}\ddot{\mathcal{N}}'^{(2)}}$$

$$\mathcal{N}'^{(1)}\big\downarrow \qquad\qquad \xrightarrow{\text{Equation (9)},\ n=2} \qquad \boxed{\mathsf{P}^{(1)}_{\cdot\cdot} \to \boxed{\mathsf{P}^{(1)} - \tfrac{1}{4c^2}\dot{\mathsf{N}}^{(1)} + \tfrac{1}{6c^2}\ddot{\Pi}^{(1)} + \tfrac{1}{96c^4}\dot{\mathsf{N}}'^{(1)}}}$$

$$\boxed{\boxed{\mathcal{M}^{(2)} + \tfrac{1}{4c^2}\ddot{\widetilde{\Lambda}}^{(2)} - \tfrac{1}{24c^2}\ddot{\mathcal{N}}^{(2)}_{(\mathrm{sym})}}}$$

In this scheme the following notations are used:

$$\mathsf{N}_{ijk} = \frac{4}{5}\int_{\mathcal{D}} \xi_i \xi_j [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \boldsymbol{j})]_k \, \mathrm{d}^3\xi \tag{B.1}$$

$$\Lambda_{ij} = \frac{1}{28}(\mathsf{M}_{qqij} + \mathsf{M}_{qqji}) \qquad \mathsf{N}_{ik} = \frac{3}{4}\int_{\mathcal{D}} \xi_i [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \boldsymbol{j})]_k \, \mathrm{d}^3\xi \tag{B.2}$$

$$\Lambda_i = \frac{1}{15}\mathsf{M}_{qqi} \qquad \mathsf{N}_i = \frac{2}{3}\int_{\mathcal{D}} [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \boldsymbol{j})]_i \, \mathrm{d}^3\xi \tag{B.3}$$

$$\Pi_{ijk} = \frac{1}{9}\mathsf{P}_{qqijk} - \frac{1}{9 \times 14}\sum_{D(i)} \delta_{ij}\mathsf{P}_{qqppk} \tag{B.4}$$

$$\Pi_{ij} = \left[\frac{1}{7}\mathsf{P}_{qqij} - \frac{1}{70}\delta_{ij}\mathsf{P}_{qqpp}\right] \tag{B.5}$$

$$\mathcal{N}_{ik} = -\frac{4}{5}\int_{\mathcal{D}} \xi^2 \xi_i \, (\boldsymbol{\xi} \times \boldsymbol{j})_k \, \mathrm{d}^3\xi \qquad \mathcal{N}_i = -\frac{3}{4}\int_{\mathcal{D}} \xi^2 (\boldsymbol{\xi} \times \boldsymbol{j})_i \, \mathrm{d}^3\xi$$

$$\widetilde{\Pi}_{ij} = \Pi_{ij} - \frac{1}{3}\delta_{ij}\Pi_{kk} \tag{B.6}$$

$$\Lambda'_k = \frac{1}{15}\ddot{\mathcal{N}}_{qqk} = -\frac{1}{18}\int_{\mathcal{D}} \xi^4 (\boldsymbol{\xi} \times \ddot{\boldsymbol{j}})_k \, \mathrm{d}^3\xi \tag{B.7}$$

$$\mathsf{N}'_k = -\frac{4}{5}\int_{\mathcal{D}} \xi^2 [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \ddot{\boldsymbol{j}})]_k \, \mathrm{d}^3\xi \qquad \Pi_i = \frac{1}{5}\mathsf{P}_{qqi} - \frac{1}{80c^2}\dot{\mathsf{N}}_{qqi} + \frac{3}{700c^2}\ddot{\mathsf{P}}_{qqppi} \tag{B.8}$$

and

$$\widetilde{\mathsf{N}}_{ijk} = \mathsf{N}_{((\mathrm{sym}))ijk} - \frac{1}{15}\sum_{D(i)} \delta_{ij}\mathsf{N}_{qqk}$$

$$\widetilde{\Pi}_{ijk} = \Pi_{ijk} - \frac{1}{5}\sum_{D(i)} \delta_{ij}\Pi_{qqk} \tag{B.9}$$

$$\widetilde{\mathsf{N}}_{ij} = \tfrac{1}{2}(\mathsf{N}_{ij} + \mathsf{N}_{ji}) \qquad \widetilde{\Pi}_{ij} = \Pi_{ij} - \tfrac{1}{3}\delta_{ij}\Pi_{qq}.$$

For $\mu < 4$ and $\varepsilon < 5$, the quantities N, Λ, ... are obtained from the above expressions by eliminating the contributions of the magnetic multipole tensors of ranks $n_m > \mu$ and of the electric ones for ranks $n_e > \varepsilon$.

We point out that the reduction scheme described in this appendix is valid also in the case of an arbitrary electromagnetic field as may be seen from [7].

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